W-Algebras *

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W-algebras are defined as polynomial extensions of the Virasoro algebra by primary fields, and they occur in a natural manner in the context of two-dimensional integrable systems, notably in the KdV and Toda systems. Their occurrence in those theories can be traced to their being the residual symmetry algebras when certain first-class constraints are placed on Kac-Moody algebras. In particular, their occurrence in 2-dimensional Toda theories is explained by the fact that the Toda theories can be regarded as constrained Wess-Zumino-Novikov-Witten (WZNW) theories. The general form of such first-class constraint for WZNW theories is investigated, and is shown to lead to a wider class of two-dimensional integrable systems, all of which have W-algebras as symmetry algebras.

1. Introduction

The scientific achievements of George Sudarshan are well-known and have been well-documented at this conference. But there is one of his achievements that is not so easy to document, and I should like to take this opportunity to draw attention to it. This is his achievement in inspiring many generations of young physicists and launching them on their careers. The gift of inspiring young people is relatively rate, although there are, of course, a number of famous exceptions such as Sommerfeld, Bethe, Schwinger, and Bob Marshak. George had this gift in great abundance, and his career has been remarkable for the manner in which he used it. In spite of his enormous output, he has always found time to inspite, instruct and encourage young physicists, and indeed his older colleagues too. There are many of us here who have benefitted from this gift of George's and I should like to take this opportunity to thank him on behalf of all of us.

One of George's great interests has been the theory of constrained systems as described in his and Mukunda's well-known book on classical mechanics. Although I worked with George when he was writing this book, I was not interested in constrained systems at the time. However, recently I have seen the light, because, it turns out that in the theory of two-dimensional integrable systems constraints play a central role.

As you all know, in recent times two-dimensional conformally-invariant field theories have attracted an enormous amount of attention. This is partly due to the fact they relate a number of hitherto unrelated physical and mathematical disciplines such as the theories of statistical mechanics, strings and Riemannian surfaces. Furthermore, they relate these subjects to the theory of integrable systems, and it is in this relationship that the theory of constraints turns out to play a central role. Indeed, as we shall see, some of the more celebrated two-dimensional systems are nothing but constrained versions of a single, trivially integrable system, the Wess-Zumino-Novikov-Witten (WZNW) one.

The versatility of two-dimensional conformal field theories rests on the fact that the two-dimensional conformal group is much less trivial than its higher-dimensional counterparts, its generators comprising *all* the components of the energy momentum tensor density and thus incorporating all of the basic physical information. As is well-known, in two dimensions the energy momentum tensor density $T_{\mu\nu}(\mathbf{x})$ has only three components, which may be written as

$$T_{++}(\mathbf{x}), \quad T_{--}(\mathbf{x}) \quad \text{and} \quad T_{+-}(\mathbf{x}), \tag{1.1}$$

where $x_\pm = x_1 \pm x_2$ and $x_1 \pm i x_2$ for the Minkowskian and Euclidean versions, respectively. For the theory to be conformally invariant there is an additional condition $T_{+-}(\mathbf{x}) = 0$. For translationally-invariant theories the two remaining components are

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chiral since the momentum conservation equations

$$\partial_\mu T_{\mu\nu}(\mathbf{x}) = 0 \Rightarrow \partial_+ T_{--}(\mathbf{x}) = 0 \quad \text{and} \quad \partial_- T_{++}(\mathbf{x}) = 0\,, \tag{1.2}$$

and thus $T_{++}(\mathbf{x})$ and $T_{--}(\mathbf{x})$ depend only on x_+ and x_-, respectively. As a matter of fact, scale-invariance alone is sufficient to ensure that $T_{+-}(\mathbf{x}) = 0$, since this is just the condition for the conservation $\partial_\mu j^\mu(x) = 0$ of the current $j^\mu(x) = x_\nu T^{\nu\mu}(x)$ that generates scale-transformations. But it is well-known that in the context of local Lagrangian field theories scale invariance usually implies conformal invariance. It should also be recalled in passing that in the quantized theory the condition $T_{+-}(\mathbf{x}) = 0$ may have to be maintained by the addition of an 'improvement' term $\Delta T_{+-}(\mathbf{x})$.

The conformal group in two dimensions consists of *all* analytic transformations $x_+ \to f(x_+)$ and $x_- \to g(x_-)$, where $x_\pm = x \pm t$ (or $x \pm it$ in the Euclidean case), where x, t are the conventional space-time coordinates. Correspondingly, its Lie algebra is the direct sum of two Virasoro algebras [1] of the form

$$\{L(y), L(y')\} = \partial_y L(y)\,\delta(y-y') + L(y)\,\partial_y \delta(y-y') + c(\partial_y)^3 \delta(y-y')\,, \tag{1.3}$$

where $y = x_\pm$, the bracket is either Poisson or commutator, and c is a constant that characterizes the one-parameter central extension that is admitted by this algebra. It is the generators $L(y)$ that are identified as the components of the energy-momentum tensor density, i.e.

$$L(x_+) = T_{++}(x_+) \quad \text{and} \quad L(x_-) = T_{--}(x_-)\,. \tag{1.4}$$

2. Examples of Two-Dimensional Conformal Field Theories

The simplest non-trivial two-dimensional conformal theory is the Liouville theory, with action

$$\int d^2x\,\{(\partial\phi(x))^2 + e^{\phi(\mathbf{x})}\}\,. \tag{2.1}$$

This action appears in a variety of situations, and in particular it is the action for the two-dimensional gravity theory induced by renormalization in standard string theory, expressed in the conformal gauge [2]. The covariant form of this action is

$$\int d^2x\,\{R(\mathbf{x})\nabla^{-1}R(\mathbf{x}) + \sqrt{g}(\mathbf{x})\}\,, \tag{2.2}$$

where $g_{\mu\nu}(\mathbf{x})$ is the metric, $R(\mathbf{x})$ the curvature and ∇ the d'Alembertian, and it reduces to (2.1) in the gauge (coordinate system) in which $g_{\mu\nu}(\mathbf{x}) = e^{\phi(\mathbf{x})}\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowskian or Euclidean metric. For a number, l say, of scalar fields the two dimensional conformal action is [3] of the form

$$I(\phi_k) = \sum_{ik} \int d^2x\,\{C_{ij}(\partial\phi_i\partial\phi_k) + g_i e^{K_{ik}\phi_k}\}\,, \tag{2.3}$$

where $i, j = 1 \ldots l$, and the C, K, and g are constants, i.e. it has an exponential-type potential.

The Liouville system (2.1) is integrable, but the system (2.3) for more than one scalar field is not, in general, integrable. On the other hand it is known that for any number of fields it becomes integrable if the g's are unity and C and K are the Coxeter and Killing matrices of any simple Lie algebra of rank l with fundamental roots α_i, i.e.

$$g_i = 1, \quad C_{ik} = \frac{4(\alpha_i,\alpha_k)}{(\alpha_i)^2(\alpha_k)^2} \quad \text{and} \quad K_{ik} = \frac{2(\alpha_i,\alpha_k)}{(\alpha_i)^2}\,. \tag{2.4}$$

Equations (2.3) and (2.4) define the Toda theories [4]. It is not clear at this level why the association of couplings in (2.3) with the C–K matrices of a simple Lie group makes the system integrable, but this is one of the questions which are answered by the WZNW reduction.

The natural generalization of the above examples to non-abelian groups is the WZNW action [1], for which the fields $g(x)$ take their values in a simple Lie group G and the action takes the form

$$I(g) = \kappa \int d^2x\,\mathrm{tr}(J_+(x)J_-(x)) + \frac{2\kappa}{3}\int d^3x\,\varepsilon_{rst}\,\mathrm{tr}(J_r(x)J_s(x)J_t(x))\,, \tag{2.5}$$

where

$$J_+(x) = g(x)\partial_+ g^{-1}(x) \quad \text{and} \quad J_-(x) = (\partial_- g^{-1}(x))\,g(x)\,. \tag{2.6}$$

Here the 3-dimensional integral is topological in the sense that is *variation* is a pure divergence and thus reduces to an integral over its boundary, which is assumed to be the 2-dimensionl space under consideration. The insertion of this term has the consequence that the field equations take ths simple form

$$\partial_- J_+(x) = 0 \quad \text{and} \quad \partial_+ J_-(x) = 0\,, \tag{2.7}$$

Which simply state that the currents $J(x)$ are chiral, i.e. are functions of $x_\pm$ only. An important consequence of the chirality is that the general solution of the field equations is simply $g(x) = g_L(x_+)g_R(x_-)$, where g_L and g_R are arbitrary (matrix-valued) functions of their arguments.

The WZNW action (2.3) is invariant with respect to the global transformations

$$g(x) \to g(x_+)\, g(x) \quad \text{and} \quad g(x) \to g(x)\, g(x_-), \tag{2.8}$$

and the Noether currents for these transformations are just the chiral currents $J_\pm(x_\pm)$. As a result, each of these currents commutes with the other one and satisfies a KM algebra of the form

$$\{J^a(y), J^b(y')\} = f_c^{ab} J^c(y))\,\delta(y-y') + \kappa g^{ab}\,\partial_y\,\delta(y-y'), \tag{3.4}$$

where the f_c^{ab} and the g^{ab} are the structure constants and Cartan metric of the semi-simple Lie group G. Thus the WZNW theories provide a natural Lagrangian realization of the KM algebra.

Since the WZNW theory is conformally-invariant, the trace $T_{+-}(x,t)$ of the WZNW energy-momentum tensor should be zero on the mass-shell, and it turns out on computation that it is actually zero both on and off the mass-shell. On the mass-shell, momentum conservation then guarantees that the remaining two components are chiral and are Virasoro operators. When written explicitly in terms of the KM currents, they turn out to be quadratic and of the form

$$L(y) = \frac{1}{(2K)}\,\mathrm{tr}\,(J(y))^2, \quad \text{where}$$
$$L(y) = T_{++}(x_+) \quad \text{or} \quad T_{--}(x_-). \tag{2.10}$$

Here $2K = 2\kappa$ for the classical version of the theory and $(2\kappa + g)$, w of G, for the quantized version.

3. W-Algebras

Fields $\phi(\mathbf{x})$ which transform covariantly with respect to the two-dimensional conformal group, i.e. according to

$$\phi(y,\tilde{y}) \to \left(\frac{\partial f(y)}{\partial y}\right)^{-s} \phi(f(y),\tilde{y}), \tag{3.1}$$

where $y = x_\pm$ and $\tilde{y} = x_\mp$, are called *primary* fields, and the (numerical) indices s are called the conformal weights. The infinitesimal form of (1.2) is seen to be

$$\int \varepsilon(x)\{L(x), \phi(y',\tilde{y})\} = \delta\phi(y,\tilde{y}) = \varepsilon(y)\,\partial_y\,\phi(y,\tilde{y}) - s\,\varepsilon'(y)\,\phi(y,\tilde{y}), \tag{3.2}$$

where $f(x) = x + \varepsilon(x)$ and $L(x)$ are the Virasoro generators.

In 1984 Zamolodchikov [5] considered the possibility that, given a Virasoro algebra with generators $L(x)$ and a finite set of primary fields $\phi_k(x)$, the Poisson brackets or commutators of the primary fields with themselves might close to yield a polynomial in the Virasoro operator, the primary fields and their derivatives. If the space-time coordinates are assigned a conformal weight (-1), in which case the delta-functions have conformal weight $(+1)$, then the polynomial is homogeneous in the weights and the brackets are of the general form

$$\{\phi_s(y)\,\phi_t(y')\} = \sum (\partial_y)^a\,\phi_u(x)\,(\partial_y)^b\,\phi_v(x)\,(\partial_y)^c\,\phi_w \dots \delta^n(x-y), \tag{3.3}$$

where $a+b+c+\dots+u+v+w+\dots+n+1 = s+t$. Such algebras are called W-algebras, and since their first proposal have been realized in a number of different situations.

The most straightforward realization of W-algebras is in the context of the KM-algebras (2.9). For these a (Poisson-bracket) W-algebra is generated by the Sugawara-Virasoro operator

$$L(y) = g_{ab}\,J^a(y)\,J^b(y) \tag{3.5}$$

(suitably normal-ordered in the quantum case) and the set of primary fields

$$W_s(y) = d_{abc\dots}\,J^a(y)\,J^b(y)\,J^c(y)\dots, \quad \text{where}$$
$$C_s = d_{abc\dots}\,X^a X^b X^c \dots \tag{3.6}$$

are the Casimir operators of order s for the generators X^a of a simple Lie group G. This was first shown by Zamolodchikov himself [5] for the SU(3) case. Whether the Poisson brackets can be generalized to commutator brackets for all representations is not yet clear [6]. (At any rate it is interesting to note that for the higher-order Casimirs no normal-ordering is necessary because the numerical d-tensors are symmetric and traceless.)

Shortly afterwards it was found that a set of Poisson-bracket algebras already considered by the mathematicians in connection with KdV hierarchies [7] are W-algebras. In a further development it was found that W-algebras were realized in a variety of Lax-pair systems [8] and in particular in Toda systems [9].

What actually happens in these cases is that the gauge-group for one of the Lax potentials $A(x) = (\partial g(x))\,g^{-1}(x)$ is a nilpotent subgroup of a semi-simple group G. Because of nilpotency, some of the elements, $e(x)$ say, of the matrix $g(x) \varepsilon G$ are gauge-invariant, and one can use the equation $\partial g(x) = A(x) g(x)$ to eliminate the other components and obtain for the gauge-invariant components higher order differential equations of the form

$$\{\partial^n + \sum W^{(r)}(A(x)) \partial^r\} e(x) = 0 . \qquad (3.7)$$

(In a few cases, this equation is actually a pseudo-differential equation, i.e. one in which there are also some inverse powers of ∂, but this makes no essential difference.) The coefficients $W^{(r)}(A(x))$ in (3.7) (or the corresponding pseudo-differential equations) are gauge-invariant, and Poisson-bracket algebra of these coefficients which is induced by the KM algebra of $A(x)$ is the W-algebra.

The present talk is primarily concerned with the Toda case. The main point is that the above occurrence of W-algebras in Toda theories, and other aspects of the Toda theories (such as their integrability), can be very easily understood by the observation [10] that Toda theory is nothing but a WZNW theory which is reduced by a set of first-class linear constraints. Using these constraints the general solution of the Toda field equations is easily deduced from the (trivial) general WZNW solution, and the W-algebras emerge as the canonical symmetry algebras of the Toda system. They are also seen to be the algebras of gauge-invariant polynomials of the constrained KM currents and to be the Dirac star-algebras of the second-class constraints produced by gauge-fixing.

4. Conformal Reduction of KM Algebras

In this section we wish to show that the WZNW theories can be reduced to the Toda theories by means of first class constraints. The form of the first-class constraints can be expressed very simply at the KM level as follows: Let the KM currents $J(y)$ of (2.9) be those in the Cartan basis, i.e. $\{J_{-\alpha}(y), J_i(y), J_\alpha(y)\}$ in conventional notation. Then the reduction is simply to let

$$J_{-\alpha_i}(y) = 1 \quad \text{and} \quad J_{-\alpha}(y) = 0 , \qquad (4.1)$$

according to the roots being fundamental or not fundamental. This reduction is first class since from (2.9) the commutation relations of any two negative components has no central term and no fundamental root. Of course, this reduction is only possible for those Lie algebras which are the *real* linear spans of the Cartan generators, the so-called real *split* Lie algebras. It is clear that these Lie algebras are highly non-compact and for each complex semi-simple Lie algebra there is just one such real form. For example, for the A and D series of Lie algebras they are the Lie algebras of $SL(N, R)$ and $SO(N, N)$, respectively.

To obtain an intuitive feeling for the meaning of the reduction (3.1) it is useful to consider the $SL(N, R)$ case, for which the reduced current takes form

$$J^{constr.}(z) = \begin{pmatrix} j_{11}(y) & j_{12}(y) & j_{13}(y) & \dots & j_{1n}(y) \\ 1 & j_{22}(y) & j_{23}(y) & \dots & j_{2n}(y) \\ 0 & 1 & j_{33}(y) & \dots & j_{3n}(y) \\ 0 & 0 & 1 & \dots & j_{4n}(y) \\ 0 & 0 & 0 & \dots & j_{5n}(y) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & j_{nn}(y) \end{pmatrix} . \qquad (4.2)$$

Although the first-class nature of the reduction (4.1) is obvious, the conformal invariance is not, since KM currents have conformal spins (± 1), and hence to put some of them equal to constants breaks the conformal invariance generated by the WZNW Virasoro algebra (2.10). So how is the conformal invariance preserved? The answer is that the Virasoro generators can be modified so that the components of the currents which are set equal to constants become scalars. The modification is

$$L(y) \rightarrow \Lambda(y) = L(y) + \partial_y H(y) , \quad \text{where}$$
$$H(y) = (H, J(x)) , \qquad (4.3)$$

and H is the (unique) element of the Cartan subalgebra for which all the fundamental roots have weight unity, $[H, E^\alpha] = E^\alpha$. It is easy to verify from the KM algebra that with respect to the conformal group generated by $\Lambda(y)$ the conformal spins of the KM current-components become $(1 + h)$, where the h are their weights with respect to H. Thus in particular the components corresponding to the negative fundamental roots $E^{-\alpha}$ (for which $h = -1$) become scalars. Setting them equal to constants then preserves the conformal invariance.

Of course, since there are two chiral sectors, a similar procedure must be carried out for each sector. So far the procedures have been chosen to be dual in the sense that for the different chiral sectors one chooses the positive and negative roots in (4.1), respectively (and makes the modification $L(y) \rightarrow \Lambda(y) = L(y) \pm \partial_y H(y)$).

The physical meaning of the field $H(y)$ is two-fold. First, in Toda theories (and their generalizations) the field $\exp(H(x_-) + H(x_+))$ can be interpreted as a 2-

dimensional gravitational connection [11]. Second, the modified Virasoro generators $\Lambda(x_\pm)$ turn out to be the components of the improved energy momentum tensor in the reduced theory.

5. Reduction of the WZNW Action

For any set of first-class constraints there is a standard strategy for obtaining the reduced action. This is to gauge the original action with respect to the group generated by the constraints (omitting kinetic terms for the gauge fields, which then appear as Lagrange-multipliers) and then to eliminate the gauge-fields by means of their Euler-Lagrange equations (or by functional integration in the quantum case).

Applying this general strategy to our case we see that the gauge groups for our constraints are the KM transformations generated by the current components $J^{\pm\alpha}(x_\pm)$), and hence the gauge fields are simply

$$A_+(x_+) = a_\alpha(x_+)E^\alpha \quad \text{and} \quad A_-(x_-) = a_{-\alpha}(x_-)E^{-\alpha}, \tag{5.1}$$

respectively. Accordingly, the gauged WZNW action is

$$I_{WZ}(g) + \int d^2x \operatorname{tr}\{A_+(J - M_-) + A_-(J - M_+) + A_+ g A_- g^{-1}\}, \tag{5.2}$$

and the Euler-Lagrange field equations for the Lagrange-multiplier fields $A_\pm$ are

$$A_+^\alpha = (E^\alpha, g(J - M_+)g^{-1}) \quad \text{and} \quad A_-^{-\alpha} = (E^{-\alpha}, g^{-1}(J - M_-)g). \tag{5.3}$$

If one re-inserts these values of $A_\pm$ in (4.2) and makes the (Gauss) decomposition

$$g = e^{\varepsilon_\alpha(x,t)E^\alpha} e^{\phi_i(x,t)H^i} e^{\varepsilon_{-\alpha}(x,t)E^{-\alpha}} \tag{5.4}$$

of g one finds that the fields $\varepsilon_{\pm\alpha}(x, t)$ drop out and (5.2) reduces to exactly the Toda action (2.3) for the fields $\phi_k(x, t)$. This derivation of the Toda theory explains why that theory is associated with the C–K matrices of a semi-simple Lie group. It also explains the integrability of the Toda theory. Indeed, the general solution of the Toda field equations can be obtained directly from the well-known general solution $g(x, t) = g_+(x_+)g_-(x_-)$ of the WZNW field equations [10].

6. W-Algebras and their Interpretation

A KM algebra such as (2.9) may be thought of as a defining a closed symplectic form and hence a phase space for the current components $J^a(y) = (a, J(y))$. In this phase space the canonical transformations generated by functionals $F(J)$ of the currents are of the form

$$\delta J^a(y) = \{F, J^a(y)\} = \int d^2y' \frac{\delta F}{\delta J^b(y')} \{J^b(y'), J^a(y)\}\,. \tag{6.1}$$

Let us now consider those functionals which preserve the constrained form of the current i.e. are such that, given

$$(E^\alpha, J^a(y)) = 0, \quad \text{we have} \quad (E^\alpha, \delta J^a(y)) = 0 \tag{6.2}$$

for the chiral sector with positive α (and similarly for the other sector). This property of preserving the form of the constrained current is evidently invariant with respect to the Poisson bracket operation and hence the set of all such functionals, which will be denoted by $W(J)$, forms a close algebra with respect to Poisson brackets. This is evidently the little algebra of the constrained currents within the canonical algebra and, as will bee seen below, that W-algebra in the sense of Zamolodchikov.

Because the W-algebras as just defined are chiral, they preserve the WZNW field equations (2.7), and since by definition they respect the constraints on the currents it follows that they preserve also the Toda equations. Thus the W-algebras emerge as symmetry algebras of the Toda system, and their Noether charges are conserved by the Toda field equations. Furthermore, it turns out that there are as many independent generators of the W-algebras as there are independent components of the Toda fields, and in this sense the W's provide a complete description of the Toda system.

A second interpretation of the W-algebras can be obtained if one recalls that the constrained components of the KM currents generate a gauge group. Then the fact that the generators W of the W-algebras commute (weakly) with the constraints, means that the W are gauge-invariant functions of the currents (and, conversely, every gauge invariant functional of the currents qualifies as a W). Hence an alternative definition of the W-algebras is as the algebras of gauge-invariant functions of the constrained currents.

Although for general reductions the bases for such algebras would not be a set of polynomial functions, the present reduction is such that they are polynomials, and thus the W-algebra is a polynomial algebra as specified by Zamolodchikov. To see this one first notes that the gauge transformations of the currents are of the form

$$J(y) \to J^g(y) = e^{a_\alpha(y)E^\alpha}(J(y) + \partial_y)e^{-a_\alpha(y)E^\alpha}\,,$$
$$\text{where} \quad J(y) = j(y) + M\,, \tag{6.3}$$

the $j(y)$ are zero on the negative root sector and $(M_-)_{rs} = \delta_{r,s+1}$. With respect to the grading operator H the current components are positive or zero, and the generators of the gauge-group are strictly positive. The crucial observation is that there exists a set of gauges (the so-called Drinfeld-Sokolov (DS) gauges [12]) in which for each positive grade only one component of the current survives (and for the zero grade none survives). Furthermore, the gauge-fixing in these gauges is complete, so the current-components calculated in these gauges constitute a complet set of gauge-invariant functions. Their polynomiality then follows from the fact that according to (6.3) the $J^g(y)$ are polynomials in the parameters $a(y)$ and their derivatives, and using $J^g(y) = J^{\mathrm{DS}}(y)$ and iterating in the grades, that the parameters themselves are polynomials in the original currents and their derivatives. The explicit details are given in [6].

A final interpretation of the W-algebras may be obtained by noting that the total set of constraints which consists of the original first-class constraints and the DS gauge-fixing, form a second-class system of constraints in the sense of Dirac. But because the ordinary and Dirac star-brackets for the functionals $W(J)$ coincide (since they respect the first-class constraints) and because the $W(J)$'s reduce to the current components in the DS gauge we have

$$\{W(J(y)), W(J(y'))\} = \{W(J(y)), W(J(y'))\}^*$$
$$= \{J^{\mathrm{DS}}(y), J^{\mathrm{DS}}(y')\}^*\,. \tag{6.4}$$

Thus we obtain a final interpretation of the W-algebras as the Dirac star-algebras of the gauge-fixed currents.

7. General Structure of Reduction and Generalizations

More recent work [13–15] concerns the analysis of the WZNW → Toda reduction with a view to simplifying and generalizing it. As the general structure is actually quite simple (in some respects simpler than the specific Toda example), I should like to conclude by sketching this structure. The general idea is to impose *linear* constraints of the form

$$J(y) = j(y) + M, \quad \text{where} \quad (\gamma, j(y)) = 0, \quad \text{for} \quad \gamma\,\varepsilon\,\Gamma\,, \tag{7.1}$$

on a KM algebra (2.9), where Γ is a subalgebra of the Lie algebra G, and M is a constant elememt of the Lie algebra which is not zero and not in Γ.

The conditions that the constraints described by (6.1) be *first-class* are two-fold, namely

$$(\alpha, \beta) = 0 \quad \text{and} \quad \omega(\alpha, \beta) = (M, [\alpha, \beta]) = 0, \quad \alpha, \beta\,\varepsilon\,\Gamma\,, \tag{7.2}$$

and follow from the fact that the KM centre κ and the constant component M of the current are not zero. The anti-symmetric form ω will be recognized as the Kostant-Kirilov (KK) form for M evaluated at the origin. It plays central role and can be used to simplify the definition of the DS gauges as follows: The extension of ω to the whole Lie algebra G vanishes on the kernel K of the operator M, but on any subspace of G complementary to K it is non-degenerate. Hence, if we assume that Γ does not intersect K we can choose a complementary space which contains Γ and an ω-dual space Θ. The DS gauges are then simply the gauges in which $(\theta, j(y)) = 0$ for all $\theta\,\varepsilon\,\Theta$.

The condition that the constraints (6.1) be *conformally, invariant* is that there should exist some grading element H in the Lie algebra G such that

$$[H, \gamma] \subset \Gamma\,, \quad (H, \gamma) = 0 \quad \text{and} \quad [H, M] = -M\,. \tag{7.3}$$

The first condition in (7.3) implies that H should be a grading operator for Γ as well as G. The most important condition is the third one which gives a specific relation between H and M. In particular it implies that the generator M is nilpotent.

Using the constraints (7.1) satisfying the first-class, and the conformal invariant conditions (7.2) and (7.3) one obtains a conformal invariant reduction of the KM system and hence (using the dual conditions for the opposite chiral sector) of any concomitant field theory, such as the WZNW theory. The reduced theory will have symmetry algebras corresponding to the W-algebras of the Toda theory, and, as before, these will be the algebras of gauge-invariant functions of the constrained currents, or, equivalently, the Dirac star-algebras of gauge-fixed currents. The only difference will be that, in general, the gauge-invariant functions will not be *polynomials* in the constrained KM currents and their derivatives.

A fairly general *sufficient* condition for the algebras to be polynomial can be found, and can be expressed quite simply in terms of the form ω. The condition is that if $\{\gamma_i, \theta_j\}$ is an H-graded basis for the complementary space spanned by $\{\Gamma, \Theta\}$ such that

$$\omega(\gamma_i, \theta_j) = \delta_{ij}, \tag{7.4}$$

then the corresponding W-algebra will be polynomial if

$$[\gamma_i, \theta_j] \subset \Gamma \quad \text{for} \quad h(\gamma_i) \geq h(\theta_j), \tag{7.5}$$

where h are the H-grades. This condition is automatically satisfied for the Toda reduction. But it can be satisfied in a variety of other cases, and in these cases provides new conformal reductions of the WZNW theories to integrable systems with polynomial W-algebras. For example it provides a generalization of the Toda system to one which consists of WZNW fields interacting in a nearest-neighbour fashion. More precisely it provides an action of the form

$$I(g_p) = \sum_p I_p(g_p) + \int d^2x \, \mathrm{tr}(g_p^{-1} M_{p,p-1} g_{p-1} M_{p-1,p}), \tag{7.6}$$

where the g_p are WZNW fields belonging to diagonal *blocks* in the original WZNW algebra, and the $M_{p,p-1}$ and $M_{p-1,p}$ are constant matrices that connect neighbouring blocks. This system reduces to the original Toda one when the blocks are 1-dimensional. It also produces the systems discussed recently in [13] and [14].

[1] P. Goddard and D. Olive, Int. J. Mod. Phys. **A 1,** 303 (1986).
[2] M. Green, J. Schwarz, and E. Witten, Superstring Theory, Cambridge University Press, 1987, A. Polyakov, Mod. Phys. Lett. **A 2** (1987).
[3] E. Braaten, T. Curtright, G. Ghandour, and C. Thorn, Phys. Lett. **125 B,** 301 (1983).
[4] A. N. Leznov and M. V. Savaliev, Comm. Math. Phys. **74,** 111 (1980).
[5] A. B. Zamolodchikov, Theor. Math. Phys. **65,** 1205 (1986); V. A. Fateev and A. B. Zamolodchikov, Nucl. Phys. **B 280,** 644 (1987).
[6] F. A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, Nucl. Phys. **304,** 348, 371 (1988).
[7] I. M. Gelfand and L. A. Dikii, Funkts. Anal Prilozhen **15** (No. 3), 23 (1981); L. A. Dickey, Commun. Math. Phys. **87,** 127 (1982).
[8] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. **A 3,** 507 (1988); S. L. Lukyanov, Funct. Anal. Appl. **22,** 255 (1989); K. Yamagishi, Phys. Lett. **205 B,** 466 (1988); P. Mathieu, Phys. Lett. **208 B,** 101 (1988); I. Bakas, Phys. Lett. **213 B,** 313 (1988); P. Di Francesco, C. Itzykson, and J.-B. Zuber, Classical W-Algebras, preprint PUTP-1211 S.Ph.-T/90–149 (1990).
[9] A. Bilal and J.-L. Gervais, Phys. Lett. **206 B,** 412 (1988); Nucl. Phys. **B 314,** 646 (1989); **B 318,** 579 (1989); O. Babelon, Phys. Lett. **215 B,** 523 (1988).
[10] J. Balog, L. Feher, P. Forgacz, L. O'Raifeartaigh, and A. Wipf, Phys. Lett. **B 227,** 214 (1990); **B 244,** 435 (1990); Ann. Physics **202,** 76 (1990).
[11] L. O'Raifeartaigh and A. Wipf, Phys. Letters **215 B,** 361 (1990); L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, Comm. Math. Phys. **141,** (1991).
[12] V. Drinfeld and V. Sokolov, J. Sov. Math. **30,** 1975 (1984).
[13] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, Ann. Phys. and DIAS Preprint in preparation.
[14] F. A. Bais, T. Tjin, and P. Van Driel, Nucl. Phys. **B 357,** 632 (1991).
[15] M. Bershadsky and H. Ooguri, Commun. Math. Phys. **126,** 49 (1989).